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Generalization of a Hardy–Littlewood–Polya Inequality

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This article is concerned with a generalization of the well-known Hardy–Littlewood–Polya (HLP) inequality to higher dimensions $n \geqslant 2$. We also show via construction of a counterexample that for certain exponents and consequently in some spaces such extension is impossible.

1. INTRODUCTION

In this paper, we generalize the well-known Hardy–Littlewood–Polya (HLP) inequality (1.1)–(1.3) to higher dimensions. This inequality has applications in many areas of mathematics including approximation of functions; see Theorem 3.4.6 in [1]. Let us recall the classical HLP inequality:

THEOREM (G. H. Hardy, J. E. Littlewood, and G. Polya [2, Theorem 330]). *Let* $1 < p < \infty$, $\varepsilon \neq p-1$. *Further, let* f *be a function defined on* $(0, \infty)$ *and such that*

$$\int_0^\infty |f(t)|^p\, t^\varepsilon\, dt < \infty.$$

Then the following inequality holds:

$$\int_0^\infty |F(t)|^p\, t^{\varepsilon - p}\, dt \leqslant \left(\frac{1}{|\varepsilon - p + 1|}\right)^p \int_0^\infty |f(t)|^p\, t^\varepsilon\, dt \tag{1.1}$$

where

$$F(t) = \begin{cases} \int_0^t |f(s)|\, ds & \text{for} \quad \varepsilon < p-1 \qquad (1.2) \\ \int_t^\infty |f(s)|\, ds & \text{for} \quad \varepsilon > p-1. \qquad (1.3) \end{cases}$$

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Our generalization of (1.1)–(1.3) to a higher dimension, $n \geqslant 2$, is

$$\int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^{k+2}} dx \leqslant \left(\frac{p}{|n-k-2|}\right)^p \int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p+2}} dx, \tag{1.4}$$

where p is a real number $\geqslant 2$, $x \in \mathfrak{R}^n$, $\nabla =$ grad, $|x| =$ Euclidean norm of x in $\mathfrak{R}^n$, and u lies in a proper function space. To show that (1.4) in fact generalizes (1.1)–(1.3) to a higher dimension, choose as u a radial function (i.e., a function of $|x|$), $\varepsilon - p = n - k - 3$, and let $r = |x|$, $w_n =$ surface area of unit sphere in $\mathfrak{R}^n$. Then we have

$$\begin{aligned}\int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^{k+2}} dx &= \int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^{n-\varepsilon+p-1}} dx = \int_0^\infty \int_{|x|=1} \frac{|u(r)|^p}{r^{n-\varepsilon+p-1}} r^{n-1}\, dw_n\, dr \\ \int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^{k+2}} dx &= w_n \int_0^\infty |u(r)|^p r^{\varepsilon-p}\, dr.\end{aligned} \tag{1.5}$$

Similarly

$$\begin{aligned}\int_{\mathfrak{R}^n} \frac{|\nabla u(x)|^p}{|x|^{k-p+2}} dx &= \int_0^\infty \int_{|x|=1} \frac{|du(r)/dr|}{r^{n-\varepsilon-1}} r^{n-1}\, dw_n\, dr \\ &= w_n \int_0^\infty \left|\frac{du(r)}{dr}\right|^p r^\varepsilon\, dr.\end{aligned} \tag{1.6}$$

Substitute (1.5) and (1.6) into (1.4) to see that (1.4) reduces to (1.1). The generalized HLP inequality (1.4) for $p = 2$ and $k = 0$ has been used in deriving a priori estimates of solutions to some nonlinear partial differential equations (see [4, 6]). One application of (1.4) in particular is to obtain estimates of the solution of semilinear Klein–Gordon equations in $\mathfrak{R}^3$

$$\Box u(t, x) + \beta u\, |u|^{p-1} (t, x) = 0, \tag{1.7}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \qquad \text{and} \qquad x \in \mathfrak{R}^3.$$

The estimate obtained then is used to prove the existence of solutions to (1.7). A solution to (1.7) can be expressed in terms of an integral equation (D'Alembert Formula)

$$u(t, x) = u_0(t, x) - \frac{\beta}{4\sqrt{2\pi}} \int_{|y-x|=t-\tau} \frac{u\, |u|^{p-1} (\tau, y)}{|y-x|} dy, \tag{1.8}$$

where the integral is taken over the backward light cone with vertex at (t, x), and u_0 is the solution of $\Box u_0(t, x) = 0$. If one uses the change of variable $Y = y - x$ and $dy = \sqrt{2}\, dY$, the integral in (1.8) becomes

$$\frac{\beta\sqrt{2}}{4\sqrt{2\pi}} \int_{|Y| \leqslant t-\tau} \frac{u\,|u|^{p-1}\,(t-|Y|)}{|Y|}\, dY. \tag{1.9}$$

One can use an inequality of type (1.4) to estimate (1.9). As an example, consider $p = 5$. Then using Holder's inequality, we obtain

$$\begin{aligned} &\left|\frac{\beta}{4\sqrt{\pi}} \int_{|Y| \leqslant t-\tau} \frac{u^5(t-|Y|, Y)}{|Y|}\, dY\right| \\ &\quad \leqslant \frac{\beta}{4\sqrt{\pi}} \|u\|_{L^\infty} \left[\int_{|Y| \leqslant t-\tau} u^6(t-|Y|, Y)\, dY\right]^{1/2} \\ &\quad \times \left[\int_{|Y| \leqslant t-\tau} \frac{u^2(t-|Y|, Y)}{|Y|^2}\, dY\right]^{1/2}. \end{aligned} \tag{1.10}$$

The first integral on the right can be approximated via an energy estimate, while the second one is of type (1.4) for $n = 3$, $p = 2$, and $k = 0$, but this can also be approximated by an energy estimate. For details and further discussion of this problem, see [6]. An estimate of the type (1.10) that leads to existence and uniqueness of the solution to the semilinear Klein–Gordon equation (1.7) motivates the search for a more general inequality

$$\left[\int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^l}\, dx\right]^{1/p} \leqslant C\left[\int_{\mathfrak{R}^n} \frac{|\nabla u(x)|^q}{|x|^s}\, dx\right]^{1/q} \tag{1.11}$$

where p, q, C, s, and l are positive real numbers. Theorems 1 and 2 show that (1.11) holds for $p = q \geqslant 2$,

$$l - s = p, \qquad C = \frac{p}{|n-l|},$$

and $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$, the set of all infinitely differentiable functions with compact support in $\mathfrak{R}^n \backslash \{0\}$, $n \geqslant 2$. For other choices of p, q, l, and C the question remains open. In Section 3 we verify, by using a counterexample, that when $l = n$, inequality (1.4) does not hold.

2. Statement of the Result

Let $n \in N$ and $k \in \mathfrak{R}$ and assume $n - k \neq 2$; then for all $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$

$$\int_{\mathfrak{R}^n} \frac{u^2(x)}{|x|^{k+2}}\, dx \leqslant \frac{4}{(n-k-2)^2} \int_{\mathfrak{R}^n} \frac{|\nabla u|^2}{|x|^k}\, dx. \tag{2.1}$$

For $2 \leqslant p < \infty$, $n-k \neq 2$, and for all $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$ we have the following generalization of (2.1):

$$\int_{\mathfrak{R}^n} \frac{|u(x)|^p}{|x|^{k+2}}\, dx \leqslant \left(\frac{p}{|n-k-2|}\right)^p \int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p+2}}\, dx. \tag{2.2}$$

This inequality will be referred to as a "generalized" Hardy–Littlewood–Polya inequality since it reduces to (1.1) when u is a radial function. This is obviously a generalization of the HLP inequality only when $p \geqslant 2$. We summarize all these in Theorem 1 and Theorem 2.

THEOREM 1. *Let $n \in N$, $k \in \mathfrak{R}$ and $n-k \neq 2$. Then for all $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$ and $2 \leqslant p < \infty$, the inequality* (2.1) *holds.*

Proof. We search for an identity,

$$\int_{\mathfrak{R}^n} \left(\frac{|\nabla u|^2}{r^k} - \frac{1}{C^2}\frac{u^2}{r^{k+2}}\right) dx = \int_{\mathfrak{R}^n} \frac{1}{h(r)} |\nabla (g(r)\, u)|^2\, dx, \tag{2.3}$$

where $r = |x|$ and g, h are two positive functions to be determined later, along with the constant C. Let

$$A = \frac{1}{h(r)} |\nabla (g(r)\, u)|^2,$$

where $u = u(x)$; then

$$\begin{aligned} A &= \frac{1}{h(r)} \left| g'(r)\, u \frac{x}{r} + g(r)\, \nabla u \right|^2 \\ &= \frac{1}{h(r)} \left\{ 2g'(r)\, g(r)\, u \left(\frac{x}{r} \cdot \nabla u\right) + g'(r)^2 u^2 + g^2(r)\, |\nabla u|^2 \right\}. \end{aligned}$$

Now consider

$$\begin{aligned} \int_{\mathfrak{R}^n} A\, dx &= \int_{\mathfrak{R}^n} \frac{2g'(r)\, g(r)}{h(r)}\, u \left(\frac{x}{r} \cdot \nabla u\right) dx \\ &\quad + \int_{\mathfrak{R}^n} \frac{g'(r)^2}{h(r)}\, u^2\, dx + \int_{\mathfrak{R}^n} \frac{g^2(r)}{h(r)}\, |\nabla u|^2\, dx. \end{aligned} \tag{2.4}$$

The first integral on the right hand side can be simplified. After using Green's identity we obtain

$$\begin{aligned} \int_{\mathfrak{R}^n} \frac{2g'(r)\, g(r)}{rh(r)}\, u \left(\sum_{i=1}^n x_i u_{x_i}(x)\right) dx &= \sum_{i=1}^n \int_{\mathfrak{R}^n} \frac{g'(r)\, g(r)}{rh(r)}\, x_i \left(\frac{\partial u^2}{\partial x_i}\right) dx \\ &= -\int_{\mathfrak{R}^n} u^2(x)\, \operatorname{div}\left(\frac{g'(r)\, g(r)\, x}{rh(r)}\right) dx. \end{aligned} \tag{2.5}$$

Use (2.4), (2.5) to expand (2.3) as

$$\int_{\Re^n}\left(\frac{|\nabla u|^2}{r^k}-\frac{1}{C^2}\frac{u^2}{r^{k+2}}\right)dx$$
$$=-\int_{\Re^n} u^2(x)\operatorname{div}\left(\frac{g'(r)\,g(r)\,x}{rh(r)}\right)dx+\int_{\Re^n}\frac{g'(r)^2\,u^2}{h(r)}\,dx$$
$$+\int_{\Re^n}\frac{g^2(r)}{h(r)}\,|\nabla u|^2\,dx. \tag{2.6}$$

The above identity holds provided

$$\frac{1}{r^k}=\frac{g^2(r)}{h(r)} \tag{2.7}$$

and

$$\frac{1}{C^2 r^{k+2}}=\operatorname{div}\left(\frac{g'(r)\,g(r)\,x}{rh(r)}\right)-\frac{g'(r)^2}{h(r)}. \tag{2.8}$$

Solving (2.7) for h and then substituting into (2.8) yields

$$\frac{1}{C^2 r^{k+2}}=\operatorname{div}\left(\frac{g'(r)\,x}{r^{k+1}g(r)}\right)-\frac{g'(r)^2}{r^k g^2(r)}. \tag{2.9}$$

Simplify (2.9) as

$$\frac{1}{C^2}\,g^2=nrgg'+r^2gg''-(k+1)\,rgg'-2r^2g'^2$$

or

$$r^2gg''+(n-k-1)\,rgg'-2r^2g'^2-\frac{1}{C^2}\,g^2=0. \tag{2.10}$$

This nonlinear O.D.E. in g has a solution of the form $g(r)=r^s$. Substituting $g(r)=r^s$ in (2.10), we see that s is a solution of the equation

$$s^2-(n-k-2)\,s+\frac{1}{C^2}=0.$$

The roots are real (which is needed to fulfill (2.5)) provided that

$$(n-k-2)^2-\frac{4}{C^2}\geqslant 0.$$

Therefore

$$\frac{1}{C^2} \leqslant \frac{(n-k-2)^2}{4} \qquad \text{or} \qquad C^2 \geqslant \frac{4}{(n-k-2)^2}.$$

And, for $C^2 = (4)/(n-k-2)$, there is a double root as $s = (n-k-2)/2$.

$$\text{Thus} \quad \begin{cases} g(r) = r^{(n-k-2)/2} \\ h(r) = r^{n-2} \\ C^2 = \dfrac{4}{(n-k-2)^2}. \end{cases} \tag{2.11}$$

This proves that (2.3) is valid when g, h, and C satisfy (2.11). Inequality (2.1) follows immediately.

THEOREM 2. *Let* $n \in N$, $k \in \mathfrak{R}$ *and* $n - k \neq 2$. *Then for all* $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$ *and* $2 \leqslant p < \infty$, *the inequality* (2.2) *holds.*

Proof. For $p = 2$ the inequality holds because of Theorem 1. Now let $p > 2$ and $\varepsilon > 0$ for $u \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$, and let $v(x) = |u^2(x) + \varepsilon|^{p/4} \chi(x)$, where $\chi(x) \equiv 1$ on the support of u and $\chi \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$. Thus $v \in C_0^\infty(\mathfrak{R}^n \backslash \{0\})$ so that it satisfies (2.1) and

$$\int_{\mathfrak{R}^n} \frac{v^2(x)}{|x|^{k+2}} \, dx \leqslant \frac{4}{(n-k-2)^2} \int_{\mathfrak{R}^n} \frac{|\nabla v(x)|^2}{|x|^k} \, dx \tag{2.12}$$

as well.

Expanding

$$|\nabla v(x)|^2 = \left| \frac{p}{4} \, |u^2 + \varepsilon|^{(p/4)-1} \, 2u(\nabla u) \, \chi(x) + |u^2 + \varepsilon|^{p/4} \, \nabla \chi \right|^2$$

and using the orthogonality of the vectors ∇u and $\nabla \chi$, $\nabla u \cdot \nabla \chi = 0$, we arrive at

$$|\nabla v(x)|^2 = \frac{p^2}{4} \, |u^2 + \varepsilon|^{(p/2)-2} \, u^2 \chi^2 \, |\nabla u|^2 + |u^2 + \varepsilon|^{p/2} \, |\nabla \chi|^2.$$

Therefore

$$\int_{\mathfrak{R}^n} \frac{|\nabla v(x)|^2}{|x|^k} \, dx = \frac{p^2}{4} \int_{\mathfrak{R}^n} \frac{|/u^2 + \varepsilon|^{p/2-2} \, u^2 \, |\nabla u|^2}{|x|^k} \, dx + \int_{\mathfrak{R}^n} \frac{|u^2 + \varepsilon|^{p/2}}{|x|^k} \, |\nabla \chi|^2 \, dx.$$

Now let $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \int_{\mathfrak{R}^n} \frac{|\nabla v(x)|^2}{|x|^k}\, dx = \frac{p^2}{4} \int_{\mathfrak{R}^n} \frac{|u|^{p-2}\, |\nabla u|^2}{|x|^k}\, dx, \tag{2.13}$$

and

$$\lim_{\varepsilon \to 0} \int_{\mathfrak{R}^n} \frac{v^2(x)}{|x|^{k+2}}\, dx = \int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx. \tag{2.14}$$

Substituting in (2.12) yields

$$\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \leqslant \frac{4}{(n-k-2)^2} \left(\frac{p^2}{4}\right) \int_{\mathfrak{R}^n} \frac{|u|^{p-2}\, |\nabla u|^2}{|x|^k}\, dx. \tag{2.15}$$

We use Holder's inequality on the right-hand side to show that

$$\int_{\mathfrak{R}^n} \frac{|u|^{p-2}\, |\nabla u|^2}{|x|^k}\, dx \leqslant \left(\int_{\mathfrak{R}^n} \frac{|u|^{q(p-2)}}{|x|^{k_1 q}}\, dx \right)^{1/q} \left(\int_{\mathfrak{R}^n} \frac{|\nabla u|^{2t}}{|x|^{k_2 t}}\, dx \right)^{1/t}, \tag{2.16}$$

where $t = p/2$, $1/q + 1/t = 1$, and $k_1 + k_2 = k$. Now let $k_1 q = k + 2$. Then $k_1 = (k+2)/q$ and $k_2 = k - (k+2)/q$. Also $q(p-2) = p$, since $1/q = 1 - 2/p$. Therefore, it is easy to establish that $k_2 t = k - p + 2$. Substitution into (2.16) gives

$$\int_{\mathfrak{R}^n} \frac{|u|^{p-2}\, |\nabla u|^2}{|x|^k}\, dx \leqslant \left(\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \right)^{1/q} \left(\int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p-2}}\, dx \right)^{2/p}, \tag{2.17}$$

which is in turn substituted in (2.15) as

$$\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \leqslant \frac{p^2}{(n-k-2)^2} \left(\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \right)^{1/q} \left(\int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p+2}}\, dx \right)^{2/p}$$

or

$$\left(\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \right)^{1-1/q} \leqslant \frac{p^2}{(n-k-2)^2} \left(\int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p+2}}\, dx \right)^{2/p}.$$

But $1 - 1/q = 1/t = 2/p$ so that we have

$$\left(\int_{\mathfrak{R}^n} \frac{|u|^p}{|x|^{k+2}}\, dx \right)^{2/p} \leqslant \frac{p^2}{(n-k-2)^2} \left(\int_{\mathfrak{R}^n} \frac{|\nabla u|^p}{|x|^{k-p+2}}\, dx \right)^{2/p},$$

which proves (2.2).

3. Construction of a Counterexample

Here we show that when $n = k + 2$, there is a sequence $u_m \in C_0^\infty(\Re^n \setminus \{0\})$ such that as $m \to \infty$,

$$\int_{\Re^n} \frac{u_m^2(x)}{|x|^{k+2}}\, dx \to \infty,$$

while

$$\int_{\Re^n} \frac{|\nabla u_m(x)|^2}{|x|^k}\, dx < \infty.$$

This shows that when $n = k + 2$, an inequality similar to (2.1) does not exist. We consider $u_m(x)$ to be a radial function; that is, $u_m(x) = u_m(r)$. Then

$$\int_{\Re^n} \frac{u_m^2(x)}{|x|^n}\, dx = w_n \int_0^\infty \frac{u_m^2(r)}{r}\, dr,$$

where w_n = surface area of unit sphere in $\Re^n$. Similarly,

$$\int_{\Re^n} \frac{|\nabla u_m(x)|^2}{|x|^{n-2}}\, dx = w_n \int_0^\infty r u_m'(r)^2\, dr.$$

The heart of the construction is the fact that the function

$$g(x) = \frac{1}{\sqrt{\ln x}}$$

satisfies

$$\int_2^\infty \frac{g(x)^2}{x}\, dx = +\infty$$

and

$$\int_2^\infty x g'(x)^2\, dx < \infty.$$

Choose $f \geq 0$, $f \in C_0^\infty(\Re)$, so that $f = 0$ on $(-\infty, 1]$ and

$$f(x) = \frac{1}{\sqrt{\ln x}} \qquad \text{for} \quad x \geq 2.$$

Then

$$\int_0^\infty \frac{f(x)^2}{x}\,dx = +\infty$$

$$\int_0^\infty xf'(x)^2\,dx < \infty.$$

Also, choose $\Psi \in C_0^\infty(\mathfrak{R})$ with $\operatorname{supp}\Psi = [-2, 2]$, Ψ equal to one on $[-1, 1]$, and $0 \leqslant \Psi \leqslant 1$ elsewhere.

Let $u_m(x) = \Psi(x/m) f(x)$. Then

$$\int_0^\infty \frac{u_m^2(x)}{x}\,dx \geqslant \int_2^m \frac{f(x)^2}{x}\,dx \to \infty \qquad \text{as} \quad m \to +\infty.$$

On the other hand

$$u_m'(x) = \frac{1}{m}\,\Psi'\left(\frac{x}{m}\right) f(x) + \Psi\left(\frac{x}{m}\right) f'(x)$$

$$|u_m'(x)| \leqslant \frac{1}{m}\,|\Psi'|_{L^\infty(\mathfrak{R})} + |f'(x)|$$

or

$$|u_m'(x)| \leqslant \frac{C}{m} + |f'(x)|$$

Using the inequality $ab \leqslant \frac{1}{2}(a^2 + b^2)$ we find that

$$u_m'(x)^2 \leqslant D\left(\frac{1}{m^2} + f'^2\right).$$

Then since u_m vanishes for $n \geqslant 2m$, we have

$$\int_0^\infty xu_m'(x)^2\,dx \leqslant \int_0^{2m} \frac{D}{m^2}\,x\,dx + D\int_0^\infty xf'(x)^2\,dx < \infty.$$

The construction is complete.

Remark 1. From the construction of identity (2.3) it is clear that

$$C = \frac{4}{(n-k-2)^2}$$

is the best possible constant for (2.1) and consequently

$$C=\left(\frac{p}{|n-k-2|}\right)^p$$

is the best for (2.2). Note that this is consistent with the best possible constant of the Hardy–Littlewood–Polya inequality.

Remark 2. The HLP inequality has an application in weighted Sobolev spaces. See for example [3, p. 28]. The generalized HLP inequality has applications, as we showed in our introduction, in establishing the existence of a solution to nonlinear Klein–Gordon equations (see [6]) and in elliptic equations (see [4, p. 451]).

Remark 3. The inequality (2.2) holds only when $2 \leqslant p < \infty$. The question of whether this can be extended to $1 < p < \infty$ is an open question.

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